503/650 2016 | Assignment 1: Math review **Revise and resubmit: yes no**

Your name: **Name of TA:**

> **Use lined papers for your assignment. Work neatly and be precise. If you use ink, no cross-outs are allowed. Staple if there are multiple pages. Put your name in the top right corner.**

This review covers the basic mathematics you will use in this class. Each section begins with examples illustrating one way to solve the problem. You are to answer the Exercises. If you find this assignment to be trivial, just do it quickly. If you are having difficulties, find a text in basic algebra and review the appropriate topics. Then, come see me or a TA for further help. You will be allowed to "revise and resubmit" this assignment until you are able to answer all questions correctly.

Part 1. Products and Quotients of Powers

Examples

1. $2^2 \times 2^4 = [2 \times 2][2 \times 2 \times 2 \times 2] = [2 \times 2 \times 2 \times 2 \times 2 \times 2] = 2^{2+4} = 2^6$
2. $e^3 e^4 = [e \times e \times e][e \times e \times e \times e] = [e \times e \times e \times e \times e \times e \times e] = e^{3+4} = e^7$
3. $a^M a^N = [a \cdots (M) \cdots a][a \cdots (N) \cdots a] = [a \cdots (M+N) \cdots a] = a^{M+N}$
4. $\dfrac{e^5}{e^3} = \dfrac{e \times e \times e \times e \times e}{e \times e \times e} = e^{5-3} = e^2$
5. $\dfrac{a^M}{a^N} = \dfrac{[a \cdots (M) \cdots a]}{[a \cdots (N) \cdots a]} = a^{M-N}$

Exercises: Show the steps to solve each problem.

1. $2^5 \times 2^5 = 2^{10}$
2. $\dfrac{7^{19}}{7^{17}} = 7^2$
3. $e^9 \times e^{-3} = e^6$
4. $\dfrac{\beta^6 \beta^7 \beta^8}{\beta^2 \beta^3 \beta^4} = \beta^{12}$
5. $\dfrac{x^N}{x^k} = x^{N-k}$

Part 2. Natural Logarithms:

Natural logarithms and exponentials are used extensively in categorical data analysis. A key reason is that they turn multiplication into addition, and addition is easier to work with than multiplication. Here are the basic principles.

Every positive real number m can be written as $m=e^p$, where e=2.71828… If $m = e^p$, then we define ln(m) = p. Similarly for $n=e^q$ and ln(n) = q. It follows that:

$$mn = e^p e^q = e^{p+q}$$

$$\ln(m \times n) = \ln m + \ln n$$

$$\ln\left(\frac{m}{n}\right) = \ln m - \ln n$$

$$\ln\left(m^n\right) = n \ln m$$

Examples

1. $\ln 2 + \ln 3 = .693... + 1.0986... = 1.7917... = \ln(2 \times 3) = \ln(6)$
2. $\ln 2 - \ln 3 = .693... - 1.0986... = -.4054... = \ln(2/3) = \ln(.6666...)$
3. $\ln\left(e^b\right) = b \ln(e) = b \times 1 = b$
4. $\ln\frac{a}{b} + \ln\frac{b}{c} = [\ln a - \ln b] + [\ln b - \ln c] = \ln a - \ln c = \ln\frac{a}{c}$

Exercises: Show the steps to solve each problem.

1. $\ln 3 + \ln 4 = \ln 12$
2. $\ln 8 - \ln 4 = \ln 2$
3. $\ln\left(\alpha x^{\beta} \varepsilon\right) = \ln \alpha + \beta \ln x + \ln \varepsilon$
4. $\ln\frac{a}{c} - \ln\frac{b}{c} = \ln\frac{a}{b}$

Part 3. Vector Multiplication: Assume $\boldsymbol{\beta}$ is a column vector and $\mathbf{x}$ is a row vector.

Example

Let $\boldsymbol{\beta}' = (\beta_0 \quad \beta_1 \quad \beta_2)$ and $\mathbf{x} = (1 \quad x_1 \quad x_2)$, then $\mathbf{x}\boldsymbol{\beta} = \beta_0 1 + \beta_1 x_1 + \beta_2 x_2 = \beta_0 + \beta_1 x_1 + \beta_2 x_2$.

Exercises: Show the steps to solve each problem.

1. Let $\boldsymbol{\beta}' = (\alpha \quad \beta)$ and $\mathbf{x} = (1 \quad x)$, then $\mathbf{x}\boldsymbol{\beta} =$
2. Let $\boldsymbol{\beta}' = (\beta_0 \quad \beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4)$ and $\mathbf{x} = (1 \quad x_1 \quad x_2 \quad x_3 \quad x_4)$, then $\mathbf{x}\boldsymbol{\beta} =$
3. Assume you have three independent variables; you decide what the substantive variables are. For example, one might be age. Show **x, β**, and **xβ**.

Part 4. Expectation

The expectation can be thought of as the mean for the entire population.

1. For a discrete variable, the expectation can be computed as:

$$E(X) = \sum_x \Pr(X = x)x$$

2. If X and Y are random variables, and a, b and c are constants, then

$$E(a + bX + cY) = a + bE(X) + cE(Y)$$

Examples

1. The mean mixes the values of X where each value is weighted by its relative frequency. If X has values of 0 and 1 with probabilities 1/4 and 3/4, then

$$E(X) = \left(0 \times \frac{1}{4}\right) + \left(1 \times \frac{3}{4}\right) = \frac{3}{4}$$

2. Let $y_i = \alpha + \beta x_i + \varepsilon_i$. Then,

$$\begin{aligned} E(y_i) &= E\left(\alpha + \beta x_i + \varepsilon_i\right) \\ &= E\left(\alpha\right) + E\left(\beta x_i\right) + E\left(\varepsilon_i\right) \\ &= \alpha + \beta E(x_i) + E\left(\varepsilon_i\right) \end{aligned}$$

Exercises: Show the steps to solve each problem.

1. A coin is flipped. Let 1 indicate a head; 0 a tail.
   a. What is the expected value when you flip a fair coin? Answer: $1/2$.
   b. If the probability of a head is .05 and a tail is .95, what is the expected value when you flip the coin? Answer: .05.
   c. If the probability of a head is p and a tail is $1\text{-}p$, what is the expected value when you flip the coin? Answer: p
2. Forty percent of the population is healthy with an average of .5 physical limitations, 55% of the population is developing problems with an average of 1.1 limitations, and 5% of the population has severe health problems with an average of 6.2 limitations. What is the average number of limitations in the population?

Part 5. Variance of a Linear Sum

1. Let X be a random variable, and a and b be constants. Then,

$$Var(a + bX) = b^2 Var(X)$$

2. Let X and Y be two random variables with constants a and b. Then,

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y)$$

Examples

1. If $Var(X)$=4, then

$$Var(3X) = 3^2 Var(X) = 9(4) = 36$$

2. If the standard deviation of Y is 3, then $Var(Y)$=9 and $Var(2Y) = 2^2(9) = 36$.
3. If X and Y are independent, then

$$Var(2X + 3Y) = 4Var(X) + 9Var(Y)$$

Exercises: Show the steps to solve each problem. Let *s* be the number of upper body limitations a person has where *Var(s)*=3; let *t* be the number of lower body limitations a person has where *Var*(*t*)=2 .

1. Suppose that $Cov(s,t) = 1$ reflecting that the more lower body limitations you have the more upper body limitations you are likely to have. Compute $Var(s+t)$. Answer: 7
2. Suppose that $Cov(s,t) = -1$. Compute $Var(s+t)$. Answer: 3.
3. Suppose that $Cov(s,t) = 0$. Compute $Var(s+t)$.

Part 6. Rescaling Variables

We often use addition and multiplication to change the scale of a variable so that the mean becomes 0 and the variance 1. If $\mathrm{E}(x) = \mu$ and $Var(x) = \sigma^2$, then $\mathrm{E}\left(\frac{x-\mu}{\sigma}\right) = 0$ and $Var\left(\frac{x-\mu}{\sigma}\right) = 1$.

Example

Assume that $\mathrm{E}(y) = 4$ and $\mathrm{Var}(y) = 9$. Then, $\mathrm{Var}\left(\frac{y-4}{3}\right) = 1$ and $\mathrm{E}\left(\frac{y-4}{3}\right) = 0$.

Exercises: Show the steps to solve each problem.

1. Assume that $\mathrm{E}(y) = 2$ and $\mathrm{Var}(y) = 16$. Construct a variable with mean 0 and variance 1.
2. Assume that $\mathrm{E}(y) = \mu$ and $\mathrm{Var}(y) = \pi^2/3$. Construct a variable with mean 0 and variance 1. Answer: $z = (y-\mu)/(\pi/\sqrt{3})$.
3. Assume that $\mathrm{E}\ (x) =<$ day of your birth $>$ and $\mathrm{Var}(x) =<$ month of your birth $>$. Construct a variable y with mean 0 and variance 1.

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